The Anagram Formula is Uniquely Maximized

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Abstract

The anagram formula is given by

$$R = \frac{n!}{n_1!n_2!\cdots n_m!}, \quad n, n_i \in \mathbb{N}$$

with

$$n_1 + n_2 + \cdots + n_m = n$$

In this paper, it is shown that for a given m, n, there is exactly one point $p \in \mathbb{N}^m$ at which R is maximized. Specifically, we have

$$\underset{n}{\text{argmax}}\ R = \prod_{i=1}^{t_1} \left\lfloor \frac{n}{m} \right\rfloor ! \prod_{i=1}^{t_2} \left(\left\lfloor \frac{n}{m} \right\rfloor + 1 \right)!$$

A more complex proof showing the linear transformation between the linear constraint and the factorial could also be used.

What goes up must come down

The intuition is that any deviation has to be paired and thus our numerators and denominators will show a particular affinity and in particular that these products are always ≥ 1. The first priority is establishing the form, the second is showing that any deviation from the form results in a larger number.

Lemma 1 (Scaling of Factorials). *Suppose we have the product $n!$ with $n \in \mathbb{N}$. Then we can re-write this product as a product of integers less than n or equal to n or as a product of integers greater than or equal to n. In particular,*

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$$(n+k)! = (n+k-c)!\prod_{i=0}^{c-1}(n+k-i), \quad n,k,c \in \mathbb{N},\ 1 \le c \le n+k$$

$$(n-k-1)! = \frac{(n-k+c)!}{\prod_{i=0}^{c-1}(n-k+i)}, \quad n,k,c \in \mathbb{N}_0,\ c \ge 0$$

Proof. Let us begin by establishing an equivalent form of a factorial-product. Observe that

$$n! = n(n-1)! \tag{1}$$

Suppose we have some factorial $(n+k)!$. Observe that we can write this factorial as

$$(n+k)! = (n+k)(n+k-1)!$$

but

$$(n+k-1)! = (n+k-1)(n+k-2)!$$

so

$$(n+k)! = (n+k)(n+k-1)(n+k-2)!$$

but

$$(n+k-2)! = (n+k-2)(n+k-3)!$$

so (1) becomes

$$(n+k)! = (n+k)(n+k-1)(n+k-2)(n+k-3)!$$

In general, if we wish to write $(n+k)!$ with our right-hand factorial term "shifted down" by some $1 \le c \le n+k$, then we have

$$(n+k)! = (n+k-c)!\prod_{i=0}^{c-1}(n+k-i), \quad n,k,c \in \mathbb{N},\ 1 \le c \le n+k \tag{2}$$

We can also write a dual form to this, namely,

$$(n+k-c)! = \frac{(n+k)!}{\prod_{i=0}^{c-1}(n+k-1)} \tag{3}$$

Similarly, if we have some factorial $(n-k-1)!$, we can observe

$$(n-k-1)! = \frac{(n-k)!}{n-k} \tag{4}$$

but

$$(n-k)! = \frac{(n-k+1)!}{n-k+1}$$

so (4) becomes

$$(n-k-1)! = \frac{(n-k+1)!}{(n-k)(n-k+1)}$$

but

$$(n-k+1)! = \frac{(n-k+2)!}{n-k+2}$$

so (4) becomes

$$(n-k-1)! = \frac{(n-k+2)!}{(n-k)(n-k+1)(n-k+2)}$$

In general, if we wish to write our factorial $(n-k-1)!$ with our left-hand factorial term "shifted up" by some $0 \leq c$, then we have

$$(n-k-1)! = \frac{(n-k+c)!}{\prod_{i=0}^{c}(n-k+i)}, \quad n,k,c \in \mathbb{N}_0,\ c \geq 0 \tag{5}$$

□

Corollary 1.1. *Any factorial-product*

$$\underbrace{n_1!n_1!\cdots n_1!}_{k_1}\underbrace{n_2!n_2!\cdots n_2!}_{k_2}\cdots\underbrace{n_p!n_p!\cdots n_p!}_{k_p},\ k_i,n_i \in \mathbb{N}_0$$

can be written as a factorial-product

$$n_j!\prod_{\substack{i=1\\i\neq j}}^{p}\lambda(i,n_j)$$

where

$$\lambda(i,n_j) =$$

Proof. By direct application of Lemma 1 to each term. (Note that we can write 3!3!4!4! as 4*4*3!3!3!3!, this will be used in the proof. But here, need to show that the "root" form can be achieved) □

Corollary 1.2 (FIX). $\prod_{i=1}^{t_{n+k}}(n+k)! = \left[\prod_{i=1}^{t_{n+k}}(n+k-c)!\right]\prod_{i=1}^{t_{n+k}}\left[\prod_{i=0}^{c-1}(n+k-i)\right], \quad n,k,c \in \mathbb{N},\ 1 \le c \le n+k$
$\prod_{i=1}^{t_{n-k-1}}(n-k-1)! = \frac{\prod_{i=1}^{t_{n-k-1}}(n-k+c)!}{\prod_{i=1}^{t_{n-k-1}}\left[\prod_{i=0}^{c-1}(n-k+i)\right]}, \quad n,k,c \in \mathbb{N}_0$

Proof. and in general we can say for t_{n-k-1} copies of $(n-k-1)$,

$$\prod_{i=1}^{t_{n-k-1}}(n-k-1)! = \frac{\prod_{i=1}^{t_{n-k-1}}(n-k+c)!}{\prod_{i=1}^{t_{n-k-1}}\left[\prod_{i=0}^{c-1}(n-k+i)\right]}, \quad n,k,c \in \mathbb{N},\ c \ge 0 \tag{6}$$

and in general we can say for t_{n+k} copies of $(n+k)!$,

$$\prod_{i=1}^{t_{n+k}}(n+k)! = \left[\prod_{i=1}^{t_{n+k}}(n+k-c)!\right]\prod_{i=1}^{t_{n+k}}\left[\prod_{i=0}^{c-1}(n+k-i)\right], \quad n,k,c \in \mathbb{N},\ 1 \le c \le n+k \tag{7}$$

□

BEGIN OTHER THEOREM ATTEMPT

Theorem 2. *two cases - 1) something like* 3!3!3!3! *(there are no n+1 terms). 2) something like* 3!3!4!4! *(there are 2 n+1 terms). In the first case, suppose we want to achieve a second form, say* 1!2!4!5!*. In this case, we have 1 and 2 shifts up, and 1 and 1 and 2 shifts down. this is due to the constraint. we also could have had* 3!0!4!5!*, 1 and 2 shifts up, 3 and 0 shifts down.*

END OTHER THEOREM ATTEMPT

Lemma 3. *Let* $c \in \mathbb{N}$*. Then c can be written as a sum of the form*

$$c = \alpha\left\lfloor\frac{c}{m}\right\rfloor + \beta\left(\left\lfloor\frac{c}{m}\right\rfloor + 1\right), \quad \alpha, m \in \mathbb{N},\ \beta \in \mathbb{N}_0$$

for all $1 \le m \le c$*.*

Proof. By cases. In the first case suppose that $m = c$. Then we have

$$\begin{aligned} c &= \alpha\left\lfloor\frac{c}{(c)}\right\rfloor + \beta\left(\left\lfloor\frac{c}{(c)}\right\rfloor + 1\right) \\ c &= \alpha(1) + \beta(1+1) \\ &= \alpha + 2\beta \end{aligned}$$

where we can trivially select $\alpha = c$ and $\beta = 0$,

$$\begin{aligned} c &= (c) + 2(0) \\ &= c \end{aligned}$$

In the second case, suppose that $m = 1$. Then we have

$$\begin{aligned} c &= \alpha \left\lfloor \frac{c}{(1)} \right\rfloor + \beta \left(\left\lfloor \frac{c}{(1)} \right\rfloor + 1 \right) \\ c &= \alpha(c) + \beta(c+1) \end{aligned}$$

where we can trivially select $\alpha = 1$ and $\beta = 0$. Now, consider the second case of $1 < m < c$. Here we have

$$1 < \left\lfloor \frac{c}{m} \right\rfloor < c$$

so define

$$d \equiv \left\lfloor \frac{c}{m} \right\rfloor$$

So, our form is now

$$c = \alpha d + \beta(d+1)$$

Suppose we selected

$$\beta = c - md \tag{8}$$

This selection for β is clearly an integer, as products of integers yield other integers and the difference of any two integers is an integer. So how do we know that $\beta \geq 0$? Observe

$$\begin{aligned} \beta &\geq 0 \\ \implies c - md &\geq 0 \\ \implies d &\leq \frac{c}{m} \\ \implies \left\lfloor \frac{c}{m} \right\rfloor &\leq \frac{c}{m} \end{aligned}$$

which is true by definition. Lastly, suppose we choose

$$\begin{aligned}\alpha &= m - (c - md) \\ &= m - \left(c - m\left\lfloor \frac{c}{m} \right\rfloor\right) \\ &= m\left(1 + \left\lfloor \frac{c}{m} \right\rfloor\right) - c\end{aligned}$$

Clearly α is an integer for all of the same reasons β is an integer. So, observe

$$\begin{aligned}m\left(1 + \left\lfloor \frac{c}{m} \right\rfloor\right) - c \geq 1 \\ m\left(1 + \left\lfloor \frac{c}{m} \right\rfloor\right) \geq c + 1\end{aligned} \tag{9}$$

But how do we show that this inequality holds? We can split this inequality into two cases, namely, when m is a factor of c and when m is not a factor of c. Let's first consider the case of m being a factor.

If m is a factor of c, this means that

$$\begin{aligned}\left\lfloor \frac{c}{m} \right\rfloor &= \lfloor k \rfloor \\ &= k \\ &= \frac{c}{m}\end{aligned}$$

For some $k \in \mathbb{N}$. This means we can write (9) as

$$\begin{aligned}m\left(1 + \frac{c}{m}\right) &\geq c + 1 \\ \implies m &\geq 1\end{aligned}$$

which is true (recall that strictly $1 < m < c$). Now, for the second case of m not being a factor, consider the inequality

$$\left\lfloor \frac{c}{m_1} \right\rfloor \leq \left\lfloor \frac{c}{m_2} \right\rfloor < \left\lfloor \frac{c}{m_3} \right\rfloor$$

where m_1 and m_3 are consecutive factors of c and $m_1 < m_2 < m_3$. Since m_1 and m_3 are factors, we can use the result from above to write

$$\frac{c}{m_1} \leq \left\lfloor \frac{c}{m_2} \right\rfloor < \frac{c}{m_3}$$

Consider the first inequality. We can re-write m_2 as

$$m_2 = m_1 + h, \ 0 < h < m_3 - m_1$$

That is, m_2 consists of an integer part m_1 and a fractional (decimal) part h,

$$\frac{c}{m_1} \leq \left\lfloor \frac{c}{m_1 + h} \right\rfloor$$

Suppose there is some $h \in (0, m_3 - m_1)$ such that our term on the right evaluates to the next integer after $\frac{c}{m_1}$ (recalling that in fact the floor function can only map to integers). Then clearly we would have

$$\frac{c}{m_1 + h} = \frac{c}{m_3}$$
$$\implies h = m_3 - m_1$$

since $\frac{c}{m_3}$ is the next available integer after $\frac{c}{m_1}$. But this contradicts the bounds on h. Thus,

$$\frac{c}{m_1} = \left\lfloor \frac{c}{m_2} \right\rfloor$$

That is the first inequality. As for the second inequality, we can observe that this result actually implies the second one since

$$\frac{c}{m_1} < \frac{c}{m_3}$$
$$\implies \left\lfloor \frac{c}{m_2} \right\rfloor < \frac{c}{m_3}$$

□

Theorem 4 (The Number of Rearrangements is Uniquely Minimized)**.** *Let*

$$R(\mathbf{n}; m, T) = \frac{n!}{\prod_{i=1}^{m} n_i!}, \ n_i, m, n \in \mathbb{N}$$

with $1 \leq m \leq T$ *and* $a_1 + a_2 + \ldots + a_m = T$. *Then there exists exactly one point* $p \in \mathbb{N}^m$ *at which* R *is maximized. In particular,*

$$\max_{\mathbf{a}} \; R = \frac{T!}{\prod_{i=1}^{t_1} \left\lfloor \frac{T}{m} \right\rfloor ! \prod_{i=1}^{t_2} \left(\left\lfloor \frac{T}{m} \right\rfloor + 1 \right)!}$$

where

$$t_1 = m - \left(T - m \left\lfloor \frac{T}{m} \right\rfloor \right)$$
$$t_2 = T - m \left\lfloor \frac{T}{m} \right\rfloor$$

Proof. We can see that our formula essentially results in factorial-products of the form

$$\underbrace{n!n!\cdots n!}_{t_n}\underbrace{(n+1)!(n+1)!\cdots(n+1)!}_{t_{n+1}} \tag{10}$$

with

$$1 \leq n \leq T-1 \tag{11}$$
$$t_n + t_{n+1} = m \tag{12}$$
$$nt_n + (n+1)t_{n+1} = T \tag{13}$$

We assert that any deviation from this distribution results in a larger number.

Now, consider a product of the form

$$P = \underbrace{1!1!\cdots 1!}_{t_1}\cdots\underbrace{(n-1)!(n-1)!\cdots(n-1)!}_{t_{n-1}}\underbrace{n!n!\cdots n!}_{t_{n'}}\underbrace{(n+1)!(n+1)!\cdots(n+1)!}_{t_{(n+1)'}}\cdots$$
$$\cdots\underbrace{(n+2)!(n+2)!\cdots(n+2)!}_{t_{n+2}}\cdots\underbrace{T!T!\cdots T!}_{t_T}$$

with

$$n \geq 1 \tag{14}$$
$$t_1 + t_2 + \ldots + t_T = m \tag{15}$$
$$1t_1 + 2t_2 + \ldots Tt_T = T \tag{16}$$

This product can be written compactly as

$$P = \left(\prod_{i=1}^{t_1} 1!\right) \cdots \left[\prod_{i=1}^{t_{n-1}} (n-1)!\right] \left[\prod_{i=1}^{t_{n'}} n!\right] \left[\prod_{i=1}^{t_{(n+1)'}} (n+1)!\right] \left[\prod_{i=1}^{t_{n+2}} (n+2)!\right] \cdots \left(\prod_{i=1}^{t_T} T!\right) \quad (17)$$

Here we will make the assertion that any product of the form in (12) with constraints (11), (12), and (13) can be written as a product in the form of (1) times a product,

$$P = S \underbrace{n!n! \cdots n!}_{t_n} \underbrace{(n+1)!(n+1)! \cdots (n+1)!}_{t_{n-1}}$$

This is clearly seen when we consider the expansion formulas given previously in (3) and (6). For example, suppose we wanted to "shift down" our $(n+3)$ terms to $(n+1)$ terms. Then we can could write

$$\begin{aligned}
\prod_{i=1}^{t_{n+3}} (n+3)! &= \left[\prod_{i=1}^{t_{n+3}} (n+3-2)!\right] \prod_{i=1}^{t_{n+3}} \left[\prod_{i=0}^{2-1} (n+3-i)\right] \\
&= \left[\prod_{i=1}^{t_{n+3}} (n+1)!\right] \prod_{i=1}^{t_{n+3}} \left[\prod_{i=0}^{1} (n+3-i)\right] \\
&= s_1 \prod_{i=1}^{t_{n+3}} (n+1)!
\end{aligned}$$

where s_1 represents the remaining product. Analogous logic follows for the case of "shifting up" factorials of the form $(n-k-1)$. Thus, we generally have

$$P = s_1 s_2 \cdots s_u \prod_{i=1}^{t_1+t_2+\cdots+t_{n'}} n! \prod_{i=1}^{t_{(n+1)'}+t_{n+2}+\cdots+t_T} (n+1)! \quad (18)$$

Define

$$\begin{aligned}
S &\equiv s_1 s_2 \cdots s_u \\
t_n &\equiv t_1 + t_2 + \cdots + t_{n'} \\
t_{n+1} &\equiv t_{(n+1)'} + t_{n+2} + \cdots + t_T
\end{aligned}$$

and (15) becomes

$$P = S \underbrace{n!n! \cdots n!}_{t_n} \underbrace{(n+1)!(n+1)! \cdots (n+1)!}_{t_{n-1}}$$

which is the desired form.

Now, the question we must is ask ourselves is this: is the product $S \geq 1$? Establishing this fact would show that indeed our original form is that which maximizes R. To show that this is in fact the case, we appeal to our planar constraint given in (13). To do this, we will consider constraints of the form

content...

In essence, any integer above $a + 1$ can be "scaled down" to achieve $a + 1$, at the expensive of multiplying by some positive number. We simply scale as much as we need to achieve our original form.

any deviation from the form given above "comes at a price"...namely, **any term that is an increase from this form will necessarily have a larger number than its corresponding denominator**. That is, all numerator terms can be paired with a denominator term, and every single numerator term is going to be larger than its denominator term.

We need to show that each term is matched, and that each denominator is larger than its numerator. In fact, the closest they can be is "one deviation" apart.

First show that each denominator term is matched by a numerator term.

One approach would be to use a prime factorization of the top numbers vs. the prime factorization of the bottom numbers.

Let's suppose we want one of our integers to be larger than itself.

□

Let M be a multiset. That is,

$$M = \left\{ a_1^{m(a_1)}, a_2^{m(a_2)}, \ldots, a_n^{m(a_n)} \right\}$$

where a_i are elements from the underlying set $A = \{a_1, a_2, \ldots, a_n\}$ and $m(a_i)$ is the multiplicity of a_i (how many copies of a_i are in M). Let R be the multiset permutation formula,

corollary - the formula maximizes entropy

Trivially, we can show that each element l_u can be paired with some element k_v if both series consist of the same number of terms (in other words, $s = t$). But what if one of the series has more terms than the other? We will note that it suffices to show that $s = t$ always. We will actually show that these two sums are always equal and that they therefore sum to 0.

Suppose that

$$\sum_{i=i}^{t} l_i \neq \sum_{i=1}^{s} k_i$$

This means that we can write (17) as

$$\gamma + \alpha p + \beta q = C \tag{19}$$

Let us define a set X whose elements are the individual integers in the constraint,

$$X = \{x_1, x_2, \ldots, x_m\}$$

Let us define three other sets I, E_1, E_2, S where the elements of each set are defined as follows,

$$\begin{aligned} I &= \{x \in X \mid x < p\} \\ E_p &= \{x \in X \mid x = p\} \\ E_q &= \{x \in X \mid x = q\} \\ S &= \{x \in X \mid x > q\} \end{aligned}$$

Clearly these sets form a partition of X (thus their union is in fact X). Our goal here is to show that there is a bijection $f : I \to S$ such that each $s_i \in$

Define two new numbers l and u,

$$l = \sum_{x \in X} p - x - 1$$
$$u = \sum_{s \in S} x - q$$

In other words, l and u are the sizes of the sets of differences between values in X less than p and x greater than q, respectively. We may be asking, why is it $p - x - 1$ and not just $p - x$? Our goal can now be achieved b showing that $l = u$.

Observe that

$$1 \leq \alpha \left\lfloor \frac{C}{m} \right\rfloor \leq C \tag{20}$$

because, in particular, constraint (8) allows us to say

$$1 \leq \left\lfloor \frac{C}{m} \right\rfloor \leq C \tag{21}$$

and when we combine (13) with (10),

$$1 \leq \alpha \leq C$$

satisfying (10). But what about β? If we define

$$d = C - \alpha \left\lfloor \frac{C}{m} \right\rfloor$$

then we see immediately that

$$d \geq 0$$

In other words, we can make *alpha* and β arbitrarily small (possibly 0 in the case of β) in order to achieve a composition of the form above. We can actually show an even stronger result than (7), namely, that

$$\alpha \geq 1$$
$$\beta = C - \alpha \left\lfloor \frac{C}{m} \right\rfloor$$

There is, in fact, exactly one set of α, β such that the above is satisfied.

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Using (1) again, we have

$$n! = \frac{n(n-1)!(n+1)}{n+1}$$

but

$$n(n-1)!(n+1) = (n+1)!$$

so

$$n! = \frac{(n+1)!}{n+1}$$

then

$$n! = \frac{(n+1)!(n+2)}{(n+1)(n+2)}$$

but

$$(n+1)!(n+2) = (n+2)!$$

so

$$n! = \frac{(n+2)!}{(n+1)(n+2)}$$

In general, if we wish to write $n!$ with our right-hand factorial term "shifted up" by some $c \geq 1$, then we have

$$n! = \frac{(n+c)!}{\prod_{i=1}^{c} n+i}, \quad n,k,c \in \mathbb{N},\ 1 \leq c \leq n \tag{22}$$

—

When factorial-products are scaled, we end up with $S = \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_m} = \frac{a_1 + a_1 + \cdots + a_1 + a_2 + \cdots}{b_1 + b_1 + \cdots + b_1 + b_2 + \cdots}$

—

Lemma 5 (Reformulations of Compositions)**.** *Let* $1 \leq n_1 \leq n_2,\ n_1, n_2 \in \mathbb{N}$ *and let the* k*-compositions of* n_1 *and* n_2 *be defined as follows,*

$$a_1 + a_2 + \cdots + a_{k_1} = n_1, \quad a_i \in \mathbb{N},\ 1 \leq k_1 \leq n_1$$
$$b_1 + b_2 + \cdots + b_{k_2} = n_2, \quad b_i \in \mathbb{N},\ 1 \leq k_2 \leq n_2$$

Then n_2 can be written as a k_1-composition,

$$b_1 + b_2 + \cdots + b_{k_1} = n_2, \quad 1 \leq k_1 \leq n_2$$

and therefore a bijection $f : A \to B$ where

$$A = \{a_1, a_2, \ldots, a_{k_1}\}$$
$$B = \{b_1, b_2, \ldots, b_{k_1}\}$$

must necessarily exist.

Proof. The bounds on k are justified by considering the binomial coefficient associated with k-compositions.[1] In other words, n can be decomposed into k terms whenever k and n are valid inputs to the binomial coefficient $\binom{n-1}{k-1}$ with the condition that k and n be positive.[2]

To show the claim, it suffices to show that k_1 is always less than or equal to k_2 since, in that case, k_2 can be made to equal k_1. Specifically, we need only show that the maximum value of k_1 is less than or equal to the maximum value of k_2. But this is readily shown since

$$\max \ k_1 \overset{?}{\leq} \max \ k_2 \tag{23}$$
$$\implies n_1 \leq n_2 \tag{24}$$

which follows from the first condition in the lemma definition. Since transforming the k_2-composition to a k_1-composition results in two compositions with the same number of terms, then we can state that there is a bijection $f : A \to B$ where

$$A = \{a_1, a_2, \ldots, a_{k_1}\}$$
$$B = \{b_1, b_2, \ldots, b_{k_1}\}$$

□

Lemma 6 (Equivalent Forms of Constraints)**.** *Let there be a constraint. Then it can be written in a certain form.*

Proof. First, consider that a constraint is a **composition**.[3] In general, a positive integer C can be composed of m positive integers as a sum of the form

[1] https://en.wikipedia.org/wiki/Composition_(combinatorics)#Number_of_compositions

[2] ibid.

[3] This step holds because compositions are partitions, and a partition of a discrete set with cardinality n can consist of at most n singleton subsets. This means that a composition can consist of at most n 1s. See the first example at https://en.wikipedia.org/wiki/Composition_(combinatorics) and also the page at https://en.wikipedia.org/wiki/Partition_(number_theory)

$$C = \alpha \left\lfloor \frac{C}{m} \right\rfloor + \beta \left(\left\lfloor \frac{C}{m} \right\rfloor + 1 \right) + \lambda_1 + \lambda_2 + \ldots + \lambda_n, \quad \beta, \lambda_i \in \mathbb{N}_0, \quad \alpha, m, C \in \mathbb{N} \tag{25}$$

with

$$1 \leq m \leq C \tag{26}$$
$$\alpha + \beta = m \tag{27}$$

For simplicity, we will write (7) as

$$C = \alpha p + \beta q + \lambda_1 + \lambda_2 + \cdots + \lambda_n \tag{28}$$

where

$$p = \left\lfloor \frac{C}{n} \right\rfloor$$
$$q = \left\lfloor \frac{C}{n} \right\rfloor + 1$$
$$n \in \mathbb{N}_0$$

We will show shortly that n must be a multiple of 2, which is crucial to our proof.

Let us first write our constraint in an equivalent form,

$$a_1 + a_2 + \cdots + a_s + \alpha' p + \beta' q + b_1 + b_2 + \cdots + b_t = C \tag{29}$$

where

$$a_i = p - k_i$$
$$b_i = q + l_i$$
$$s + \alpha' = \alpha$$
$$t + \beta' = \beta$$

Notice that we have not changed the number of terms in the original constraint. Rather, we have "labeled" each term according to whether or not it is less than p, equal to p, equal to q, or greater than q. Suppose we re-write (14) as

$$(p - k_1) + (p - k_2) + \cdots + (p - k_s) + \alpha' p + \beta' q + (q + l_1) + (q + l_2) + \cdots + (q + l_t) = C \tag{30}$$

where

$$p - k_i = a_i$$
$$l_i - q = b_i$$

but then (15) is readily seen to be

$$\sum_{i=i}^{t} l_i - \sum_{i=1}^{s} k_i + sp + \alpha' p + \beta' q + tq = C \tag{31}$$

$$\implies \sum_{i=i}^{t} l_i - \sum_{i=1}^{s} k_i + \alpha p + \beta q = C \tag{32}$$

where (17) is seen to be of the form of (13), as desired. In other words, we can "scale down" and "scale up" integers greater than q and less than p, respectively. Now, we wish to show that it is possible to pair each element in the first series of (17) with each element of the second series of (17). Note that this notion is not the same as showing that the series sum to some similar number. We are only concerned with showing that *there is a pairing of the elements of each series with the other series.* But how do we show this?

First, let us consider the case where the two sums are equal (but don't necessarily have the same number of terms),

$$\sum_{i=i}^{t} l_i = \sum_{i=1}^{s} k_i$$

Re-write the sums as t- and s-compositions and observe the (non-strict) inequality,

$$\sum_{i=i}^{t} l_i = \sum_{i=1}^{s} k_i \tag{33}$$

$$\implies l_1 + l_2 + \cdots + l_t \leq k_1 + k_2 + \cdots + k_s \tag{34}$$

$$\implies 1 \leq n_1 \leq n_2 \tag{35}$$

By Lemma 1, there must be a bijection (pairing) between at least one composition of n_1 and one composition of n_2.

Now we move on to the case of one sum being greater than the other (it does not matter if the lesser or the greater terms are chosen),

$$\sum_{i=i}^{t} l_i \leq \sum_{i=1}^{s} k_i$$

Similar to above, we can re-write the inequality as a comparison of compositions,

$$\sum_{i=i}^{t} l_i \leq \sum_{i=1}^{s} k_i \quad (36)$$

$$\implies l_1 + l_2 + \cdots + l_t \leq k_1 + k_2 + \cdots + k_s \quad (37)$$

$$\implies 1 \leq n_1 \leq n_2 \quad (38)$$

where again we appeal to Lemma 1 to show the bijection (pairing).

□